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A Modification of the Nested Logistic Population Model for the Study of Classified Population

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Abstract
Discussing some limitations of the nested logistic population model, the paper presents a modification of the nested logistic population model namely classified logistic model which is seen to be more suitable for the projection of classified population. The nested logistic model is dissimilar to classical models in population dynamics. The article contains the discussion on the existence-uniqueness and stability of the classified logistic model. The paper derives the analytic solution of the model, discusses the qualitative analysis of the solution and compares the modified model with the unmodified version.

Keywords : Nested Logistic Population Model, Classified Logistic Population Model, Existence-Uniqueness, Stability.

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1. Introduction

Population is a crucial field of study for almost all countries of the world, especially for the least developed countries like Bangladesh. Over population leads many other problems like poverty, illiteracy, unemployment, lack of nutrition etc. Along with the other reasons of population growth, the increasing female population has been treated as a threat for further increase in population. Unfortunately, Bangladesh is facing that kind of problem as well. Thus we feel the necessity of a Mathematical Model which can handle the total population and the female population simultaneously.

Several studies have been conducted regarding population projection in Bangladesh. *Obaidullah (1976)* presented a model termed as 'Expo-linear Model' which is claimed better than either an exponential or a linear model in describing population growth over time. *Mallick*

(1980) studied the population trend in Bangladesh and commented that the possibility of zero population growth rate in 2080. Bashar (1985) studied population growth for the graduation of the past population and prediction of the future population. According to him, among the models studied, polynomial of degree four was found to be the best one to reproduce the past population. *Rahman (1993)* used the formula $p(t) = p_0 e^{(r+k)t}$ where p_0 = population at present and $p(t)$ = population after time t, r = growth rate and k is the annual migration rate. *Hoque, Ahmed and Sarker (1995)* used this discrete model with the modification of the constant change of the growth rate.

The widely used model in Population Dynamics was developed by Belgian mathematician *Pierre Verhulst (1838)* who suggested that the rate of population increase may be limited. In a dense situation, the population fights among themselves for limited resources which creates a negative impact in the population growth. Verhulst modified the Malthusian model mentioned in *Thomas Robart Malthus (1798)* by adding a negative quadratic term in the form

$$\frac{dp}{dt} = ap(k-p), \quad p(0) = p_0 \tag{1}$$

This is known as *logistic population model.*

We have modified the well-known logistic population model described in (1) for the study of classified population and termed it as *nested logistic population model* in *Hossain (2013).* The model was capable of handling the classified population but study shows that it yields unrealistic situations in the long run. In this work, we would like to modify the nested logistic population model for resolving the unrealistic situation.

To understand the reasons behind modification of the model, it is necessary to understand the shortcoming of our proposed model. Here we recall the nested logistic population model from *Hossain (2013)*, which is

$$\frac{dx}{dt} = ax(k-x), \quad \frac{dy}{dt} = by(x-y), \; x(0) = x_0, \; y(0) = y_0 \tag{2}$$

Where, the two sets X and Y are such that $Y \subseteq X$ and each of which has a separate logistic growth and the sizes of X and Y are measured in terms of x and y respectively. The Model has the following analytic solution-

$$\left(\begin{array}{l} \left\{x[t] \to \left(1 + \frac{1}{(-1+e^{k(at+C[1])})}\right) k\right\}, \\ \left\{y[t] \to \frac{a\left(1-e^{-k(at+C[1])}\right)^{\frac{b}{a}} \left(1-e^{k(at+C[1])}\right)^{\frac{b}{a}} k}{-b\left(-1+e^{-k(at+C[1])}\right)^{\frac{b}{a}} Beta\left[e^{-k(at+C[1])}, -\frac{b}{a}, \frac{a+b}{a}\right] + a\left(1-e^{-k(at+C[1])}\right)^{\frac{b}{a}} kC[2]}\right\} \end{array}\right)$$

By mere inspection we have, $\lim_{t\to\infty} x[t]=k$. To obtain the limiting value of $y[t]$ we recall the following graphical representation of $x[t]$ and $y[t]$.

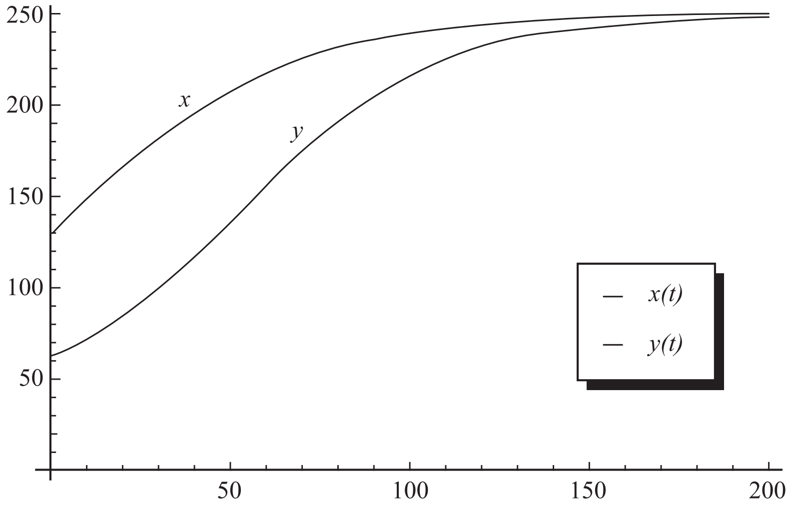

Fig-1 : Graph indicating the behavior of the solution of the model.

From the graph it is quite clear that $\lim_{t\to\infty} y[t]=\lim_{t\to\infty} x[t]=k$. This is the actual point for which the modification of our model is required. Because in representing the total population by $x[t]$ and the female population by $y[t]$, our model results in a zero male population in the long run, this is not realistic.

The actual source of unrealistic situation is that we have modeled only the growth of female and the total population. Considering so; the growth of male population was ignored automatically. Moreover, the growth of female population under the carrying capacity, x (total population) tends to occupy the full size of the carrying capacity k. As a result of which the male population has been forced to decrease by the female population and finally results in a zero in the male population.

Thus we feel the necessity of modifying the nested logistic model in *Hossain (2013)*. In this work we have modified the model in section 2 and term it as the classified logistic model. We have investigated the existence and uniqueness of the solution in section 3. The Analytic solution of the proposed new model is presented and depicted in a graphical context in section 4, to ensure the elimination of the unrealistic situation. The stability on initial and equilibrium point is also investigated in section 5.

2. The Classified Logistic Population Model

Let us consider three sets X, Y and Z representing the total, female and male population respectively as represented in the following figure.

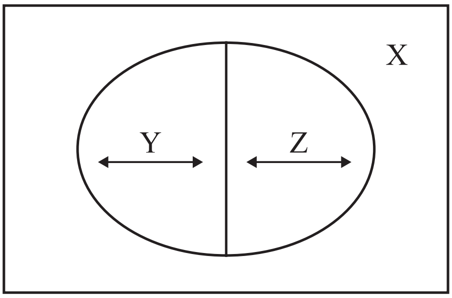

Fig-2 : The Classified Population

The figure above also contain a set *K* representing the maximum available size of the set *X* i.e. the total population which may termed as the carrying capacity of *X*. Clearly, *X*, *Y* and *Z* are increasing sets whereas the set *K* is fixed in size. Also, $X = Y \cup Z$ and $Y \cap Z = \phi$. Thus the sets *Y* and *Z* are two partitions/classes of the set *X*. Hence if size of the sets *X, Y* and Z are measured in terms of x, y and z respectively then

$$x = y + z \tag{3}$$

Here we assume the set *X* has a logistic growth *i.e.* the growth of *X* merely depends on its current size *(x)* and on the remaining size *(k-x)*, where k represents the size of the set *K*. Hence we obtain

$$\frac{dx}{dt} = ax(k-x) \tag{4}$$

where *a* is a proportionate constant.

Now, differentiating (3) with respect to time, we get

$$\frac{dx}{dt} = \frac{dy}{dt} + \frac{dz}{dt} \tag{5}$$

Which clearly shows that the growth rate of the set *X* is entirely distributed between the sets *Y* and *Z*. Now it is a fair way to think this distribution is done according as the ratio of current sizes of the two sets *Y* and *Z*. Assuming so, we have,

$$\frac{dy}{dt} = \frac{dy}{dt}\frac{y}{x} \tag{6}$$

$$\frac{dz}{dt} = \frac{dx}{dt}\frac{z}{x} \tag{7}$$

Using (4) in (6), we get

$$\frac{dy}{dt} = ay(k\text{-}x) \tag{8}$$

Similarly, using (4) in (7), we get

$$\frac{dz}{dt} = az(k\text{-}x) \tag{9}$$

Hence the equations (4), (8) and (9) give the rate of changes in the sets *X, Y, Z* respectively and can be considered as our modified model for the classified population. Introducing the initial conditions, the model will collectively look like the following :

$$\left.\begin{array}{l} \frac{dx}{dt} = ax(k\text{-}x),\ \frac{dy}{dt} = ay(k\text{-}x),\ \frac{dz}{dt} = az(k\text{-}x) \\ x(0)=x_0,\ y(0)=y_0,\ z(0)=z_0 \end{array}\right\} \tag{10}$$

In this paper we term this model as the **Classified Logistic Model** according as its nature of classification of total population.

3. Existence & Uniqueness of the Solution of the Model

For the study of existence and uniqueness of a given system of autonomous ordinary differential equation Picard-Lindelof Existence-Uniqueness theorem is very useful. Here we give a more generalized form of Picard-Lindelof Existence-Uniqueness theorem. The following statement may be treated as the Picard-Lindelof existence-uniqueness theorem in[3].

Theorem-1 : The solution of a given autonomous system of ordinary differential equation

$$\frac{dx}{dt} = f\,(x, y, z),\ \frac{dy}{dt} = g\,(x, y, z),\ \frac{dz}{dt} = h\,(x, y, z)$$

exists and is unique, if the partial derivatives

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ and $\frac{\partial h}{\partial z}$ are continuous for all values of x, y and z.

Now we consider the system of differential equations :

$$\frac{dx}{dt} = ax\,(k\text{-}x),\ \frac{dy}{dt} = ay\,(k\text{-}x),\ \frac{dz}{dt} = az\,(k\text{-}x) \tag{11}$$

Comparing (11) with the following set of equations,

$$\frac{dx}{dt} = f(x,y,z),\ \frac{dy}{dt} = g(x,y,z),\ \frac{dz}{dt} = g(x,y,z) \tag{12}$$

we have, $f(x,y,z)=ax(k-x)=akx-ax^2$, $g(x,y,z)=ay(k-x)=aky-axy$ and h $(x,y,z) = az(k-x) = akz-axz$. Taking partial derivatives of $f(x,y,z)$, $g(x,y,z)$ and h (x,y,z), we get,

$$\frac{\partial f}{\partial x} = ak - 2ax \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial g}{\partial x} = -ay \qquad \frac{\partial g}{\partial y} = ak - ax \qquad \frac{\partial g}{\partial z} = 0$$

$$\frac{\partial h}{\partial x} = -az \qquad \frac{\partial g}{\partial y} = 0 \qquad \frac{\partial g}{\partial z} = ak - ax$$

Since all of the derivatives are polynomials in x, y and z, they are continuously 1^{st} order differentiable *i.e.* c^1 Hence by Theorem-1 the proposed problem has a unique solution.

4. Analytic Solution of the Model

We obtain the analytic solution of the classified logistic population model by Mathematica 7.0 code. The solution is presented below.

$$\{\{x[t] \to \frac{e^{akt}kx}{k - x_0 + e^{aktx_0}}\},\{y[t] \to \frac{e^{-ak\left(-t+\frac{Log\left[\frac{1-e^{akt}x_0}{-k-x_0}\right]}{ak}\right)}ky_0}{k-x_0}\},$$

$$\{z[t] \to \frac{e^{-ak\left(-t+\frac{Log\left[\frac{1-e^{akt}x_0}{-k-x_0}\right]}{ak}\right)}kz_0}{k-x_0}\}\}$$

The corresponding Mathematica 7.0 code:

system = {x'[t] = = a x[t] (k-x[t]),y'[t] = = b y[t] (k-x[t]), z'[t] = = b z[t] (k-x[t]), x[t] = = x0, y[t] = = y0, z[t] = = z[0]};
sol = DSolve[system,{x[t],y[t],z[t]},t]//FullSimplify

The graphical simulation of the model is done by the help of Mathematica 7.0 for artificial data which is shown in the following figure. The figure gives a more clear conception about the behavior of the solution of the model and the relation between them.

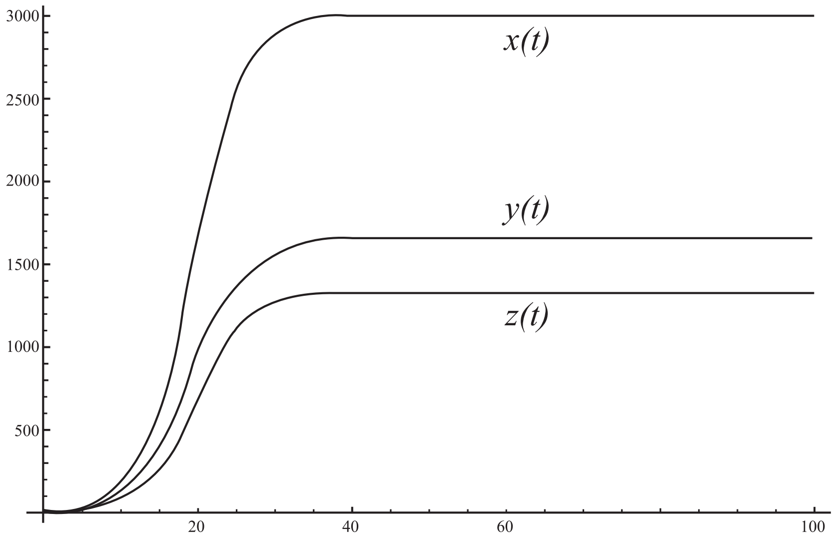

Fig-2 : Graph indicating the behavior of the solution of the model.

Fig-2 ensures the non-zeroness of the male population which was a vital point for which the modification was required for our previous model. Though the difficulties have been removed, the modified model also requires some qualitative analysis such as the continuous dependency on the initial data and the stability at the point of equilibrium. We concentrate our study on such things in few upcoming sections.

5. Stabilty

Here we refer to an important theorem which gives a confident way to examine the stability of a given autonomous system.

Theorem-2 : If λ_i be the eigen-values of a given linearized autonomous system of ordinary differential equations

$\frac{dx}{dt} = f(x, y, z),\ \frac{dy}{dt} = g(x, y, z),\ \frac{dz}{dt} = h(x, y, z)$ at a particular point, then the solution is

(1) stable if all $\lambda_i \leq 0$.

(2) asymptotically stable if all $\lambda_i < 0$.

(3) unstable otherwise.

Now we check the stability of the model at the initial point. Linearization of the model at the initial point $\{x=x_0, y=y_0, z=z_0\}$ yields the eigen-values $\lambda=a(k-2x_0), a(k-x_0), a(k-x_0)$. Since all the symbols are non-negative, the eigen-values merely depend on the value of x_0 and therefore, there may the four possible cases-

Case-I : $\{x_0 = 0\}$ In this situation the three eigen-values are $\lambda_1=ak$, $\lambda_2=ak$ and $\lambda_3=ak$, which implies $\lambda_1>0$, $\lambda_2>0$ and $\lambda_3>0$. Therefore by theorem-2, the solution is unstable.

Case-II : $\left\{0 < x_0 < \frac{k}{2}\right\}$ The three eigen-values $\lambda_1>0$, $\lambda_2>0$ and $\lambda_3>0$. Thus by theorem-2, this is an unstable solution.

Case-III : $\left\{\frac{k}{2} < x_0 < k\right\}$ The eigen-values $\lambda_1\leq 0$, $\lambda_2>0$ and $\lambda_3>0$. Thus by theorem-2, the solution is unstable in this case.

Case-IV : $\{x_0 = k\}$ In this situation the eigen-values $\lambda_1= -k < 0$, $\lambda_2= 0$ and $\lambda_3= 0$. Hence by theorem-2, the solution is stable in this case.

Now we have an interest to check the stability at the equilibrium point. the equilibrium points can be obtained by solving the equations $f(x, y, z) = 0$, $g(x, y, z) = 0$ and $h(x, y, z)=0$ simultaneously *i.e.* by solving

$ak(x - y) = 0,\ ay(k - x) = 0,\ az(k - x) = 0$

Solving these equations, we have,

$\{x = 0, y = 0, z = 0\}$, $\{x = k, y = 0, z = 0\}$, $\{x = 0, y = k, z = 0\}$, $\{x = 0, y = 0, z = k\}$, $\{x = k, y = k, z = 0\}$, $\{x = 0, y = k, z = k\}$, $\{x = k, y = 0, z = k\}$, $\{x = k, y = k, z = k\}$

As $Y \subseteq X$, $Z \subseteq X$ implies $y \leq x$, $z \leq x$ and therefore the solutions $\{x = 0, y = k, z = 0\}$,

$\{x = 0, y = 0, z = k\}$, $\{x = 0, y = k, z = k\}$ are unrealistic.

Moreover, the set Y and Z are the partitions of X. Hence $x=y+z$. Which indicates that the solutions $\{x = k, y = 0, z = 0\}$, $\{x = k, y = k, z = k\}$ are also unrealistic. Thus the equilibrium points are, $\{x=0, y=0, z=0\}$, $\{x=k, y=k, z=0\}$, $\{x=k, y=0, z=k\}$

At the point $\{x=0, y=0, z=0\}$ the eigen values are $\lambda=ak, ak, ak$. Thus by theorem-2, the solution is unstable. At the point $\{x=k, y=k, z=0\}$ we have $\lambda=0, 0, -ak$. Hence according to theorem-2, the solution is stable at $\{x=k, y=k, z=0\}$. At the point $\{x=k, y=0, z=k\}$ we have, $\lambda=0, 0, -ak$. Hence the solution is stable at $\{x=k, y=0, z=k\}$ by the same theorem.

6. Conclusions

The classified logistic population model is more suitable than the nested logistic population model for the study of classified population. But the data should be collected carefully because of the continuous independence of data of the classified growth model in some situations. The model can be generalized for higher number of classes. Therefore we are very hopeful in successful use of the model in age grouped populations. We have an intention to model age grouped population with some numerical experiments by dint of this model in some of our future work.

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