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On Some Characteristics of the Joint Distribution of Sample Variances

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Abstract

The joint distribution of correlated sample variances and their product moments have been derived. Finite expressions have been derived for product moments of sample variances of integer orders. Marginal and conditional distributions, conditional moments, coefficient of skewness and kurtosis of conditional distribution of a sample variance given the other variance have also been discussed. Shannon entropy of the distribution is also derived. When the variables are uncorrelated, the resulting characteristics match with the independent case of sample variances.

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1. Notations and Introduction

$\underset{\sim}{X}_j = \begin{pmatrix} X_{1j} \\ X_{2j} \end{pmatrix}$, $j = 1, 2, ..., N(> 2)$: Two dimensional normal random vectors,

X_{1j}, $j = 1, 2, ..., N(> 2)$: sample observations of the first component of a two- dimensional independent normal random vector, X_{2j}, $j = 1, 2, ..., N(> 2)$: sample observations of the second component of a two-dimensional independent normal random vector,

$\bar{X}_1 = \sum_{j=1}^{N} X_{1j}$, $\bar{X}_2 = \sum_{j=1}^{N} X_{2j}$: Mean of components, $\underset{\sim}{\bar{X}} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}$: Vector of sample means,

$\sum_{j=1}^{N} (X_j - \bar{X})(X_j - \bar{X})' = A$: The sums of squares and cross product (SSCP) matrix,

$A = (a_{ik}), i = 1, 2; k = 1, 2$: SSCP matrix is written by its elements,

where $a_{ii} = mS_i^2 = \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)^2$, $m = N - 1$, and $a_{12} = \sum_{j=1}^{N} (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mRS_1$

$\Sigma = (\sigma_{ik})$, $i = 1, 2$; $k = 1, 2$: Variance covariance matrix,

where $\sigma_{11} = \sigma_1^2$, $\sigma_{22} = \sigma_2^2$, $\sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0$, $\sigma_2 > 0$,

ρ $(-1 < \rho < 1)$: The product moment correlation coefficient between X_1 and X_2.

Fisher (1915) derived the distribution of the matrix A in order to study the distribution of correlation coefficient for a bivariate normal sample. The joint density function (pdf) of the elements of A can be written by the following :

$$f(a_{11}, a_{22}, a_{12}) = c\ \left(a_{11}a_{22} - a_{12}^2\right)^{\frac{m-3}{2}} \exp\left(-\sum_{k=1}^{2} \frac{a_{kk}}{2(1-\rho^2)\sigma_k^2} + \frac{\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right), \qquad (1.1)$$

where $2^m \sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)c = (1-\rho^2)^{-m/2}(\sigma_1\sigma_2)^{-m}$,

with $a_{11} > 0$, $a_{22} > 0$, $-\sqrt{a_{11}a_{22}} < a_{12} < \sqrt{a_{11}a_{22}}$, $m > 2$, $-1 < \rho < 1$ (Anderson, 2003, 123).

For the estimation of correlation coefficient by modern techniques, we refer to Ahmed (1992). The joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivariate chi-square distribution, was derived by Joarder (2007) in the spirit of Krishnaiah, Hagis and Steinberg (1963) who studied the bivariate chi-distribution. The product moment correlation coefficient between U and V can be calculated to be ρ^2. In case the correlation coefficient $\rho = 0$, the density function of U and V becomes that of the product of two independent chi-square variables each with m degrees of freedom.

In this paper, we derive the joint pdf of sample variances, namely, S_1^2 and S_2^2 in

Theorem 3.1. With the help of transformations in the density function, we derive the joint characteristic function in Theorem 3.2, the marginal distribution of variance, conditional distribution of variances, moments and some other characteristics of conditional distribution are discussed in Section 4. Product moments and Shannon entropy are discussed in Section 5 and Section 6 respectively. Correlation between sample variances is also derived in section 5. In the special case of $\rho = 0,$ the findings in the paper matches with the independent case of sample variances. The results in the paper can be extended to joint distribution of sample variances based on scale mixture of bivariate normal distributions along Joarder and Ahmed (1996).

2. Some Mathematical Preliminaries

The hypergeometric function ${}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;z)$ is defined by

$$ {}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_{\{k\}}(a_2)_{\{k\}}\cdots(a_p)_{\{k\}}}{(b_1)_{\{k\}}(b_2)_{\{k\}}\cdots(b_q)_{\{k\}}} \frac{z^k}{k!}, \tag{2.1} $$

where $a_{\{k\}} = a(a+1)\cdots(a+k-1).$

The hypergeometric function ${}_1F_1(a;b;z)$ is related to generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ by the relation

$$ L_n^{(\alpha)}(z) = \binom{n+\alpha}{n} {}_1F_1(-n;\alpha+1;z) \tag{2.2} $$

(Abramowitz and Stegun, 1964, #22.5.54, p780). Note that ${}_1F_1(a,b;z)$ can also be transformed as

$$ {}_1F_1(a;b;z) = e^z \, {}_1F_1(b-a;b;z) \tag{2.3} $$

(Abramowitz and Stegun, 505; #13.1.27). Also ${}_2F_1(a,b;c;z)$ can be transformed as

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b} \, {}_2F_1(c-a,c-b;c;z) \tag{2.4} $$

(Gradshtyen and Ryzhik, 1992, 1069). The function ${}_2F_1(a,b;c;z)$ is also related to Jacobi's polynomial $P_n^{(\alpha,\beta)}(z)$ in the following way:

$$ {}_2F_1(-n,\alpha+1+\beta+n;\alpha+1;z) = \frac{n!}{(n+1)_{\{n\}}} P_n^{(\alpha,\beta)}(1-2z). \tag{2.5} $$

We will also be using the following descending and ascending factorial of k terms:

$$ a^{\{k\}} = a(a-1)\cdots(a-k+1), \quad a^{\{0\}} = 1. \tag{2.6} $$

$$a_{\{k\}} = a(a+1)\cdots(a+k-1),\ \ a_{\{0\}} = 1. \tag{2.7}$$

3. The Joint Density Function of Sample Variances

Consider a random sample of size N represented by $\underset{\sim}{X}_j = (X_{1j}, X_{2j}, \cdots, X_{pj})', j = 1, 2, \cdots, N$ where $\underset{\sim}{X}_j \sim N_p(\underset{\sim}{\mu}, \Sigma)$, $\underset{\sim}{\mu} = (\mu_1, \mu_2, \cdots, \mu_p)'$ and $\Sigma = (\sigma_{ik})$, $i = 1, 2, \cdots, p;\ k = 1, 2, \cdots, p$ is a $p \times p$ positive definite matrix. Then $U_i = \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)^2 / \sigma^2 \sim \chi^2_m \quad (i = 1, 2, \cdots, p)$ with $m = N - 1$, and the joint distribution of $U_1, U_2, \cdots, U_p$ is called a p-variate chi-square distribution (Krishnaiah, Hagis and Steinberg, 1963). In this section we derive joint pdf of sample variances S_1^2 and S_2^2, and study some related properties in the subsequent chapters. Though the joint density function follows from Krishnaiah, Hagis and Steinberg (1963) or Joarder (2007), we provide direct derivation to provide better insight.

Theorem 3.1 The joint pdf of S_1^2 and S_2^2 is given by

$$f_2(s_1^2, s_2^2) = \left(\frac{m}{2\sigma_1\sigma_2}\right)^m \frac{(s_1^2 s_2^2)^{(m-2)/2}}{(1-\rho^2)^{m/2}\Gamma^2(m/2)} \exp\left[\frac{-m}{2-2\rho^2}\left(\frac{s_1^2}{\sigma_1^2} + \frac{s_2^2}{\sigma_2^2}\right)\right]$$

$$\times\, {}_0F_1\left(\frac{m}{2}; \frac{(m\rho)^2 s_1^2 s_2^2}{(2-2\rho^2)^2 \sigma_1^2 \sigma_2^2}\right), \tag{3.1}$$

where $m = N - 1 > 2$, $-1 < \rho < 1$ and the function ${}_0F_1(a; z)$ is defined in (2.1).

Proof. Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ in (2.1) with Jacobian $J((a_{11}, a_{22}, a_{12}) \to (s_1^2, s_2^2, r)) = m^3 s_1 s_2$, the joint pdf $f_1(s_1^2, s_2^2, r)$ of S_1^2, S_2^2 and R can be written out, and then the pdf of S_1^2 and S_2^2 can be written as

$$f_2(s_1^2, s_2^2) = \left(\frac{m}{2\sigma_1\sigma_2}\right)^m \frac{(1-\rho^2)^{-m/2}}{\sqrt{\pi}\Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})} (s_1 s_2)^{m-2} \exp\left[-\frac{m}{2(1-\rho^2)}\left(\frac{s_1^2}{\sigma_1^2} + \frac{s_2^2}{\sigma_2^2}\right)\right] I(s_1, s_2)$$

where $I(s_1, s_2) = \int_{-1}^{1} (1-r^2)^{(m-3)/2} \exp\left(\frac{m\rho s_1 s_2 r}{(1-\rho^2)\sigma_1\sigma_2}\right) dr.$

Having evaluated the last integral and made some algebraic simplifications, we have the pdf in (3.1).

It has been checked with MATHEMATICA 5.0 that the function in (3.1) integrates to 1. Figure 1 in the Appendix shows the bivariate surface of the density function in (3.1) for various values of ρ for m = 5.

Theorem 3.2 The characteristic function of the joint pdf in (3.1) of sample variances is given by

$$\phi(t_1,t_2)=\left[\left(1-\frac{2i\sigma_1^2}{m}t_1\right)\left(1-\frac{2i\sigma_2^2}{m}t_2\right)+\frac{4\sigma_1^2\sigma_2^2\rho^2}{m^2}t_1t_2\right]^{-m/2},$$

where $-1<\rho<1$ and $m>2$.

Proof. The characteristic function of the joint pdf of sample variances in (3.1) is given by $\phi(t_1,t_2)=\int_{v=0}^{\infty}\int_{u=0}^{\infty}e^{i(t_1u+t_2v)}f(u,v)dudv$. The integral can be evaluated by expanding ${}_0F_1(b;z)$ in the density function $f_2(s_1^2,s_2^2)$.

4. Marginal and Conditional Distributions

Theorem 4.1 Let S_1^2 and S_2^2 be sample variances with joint pdf given by (3.1). Then $S_i^2(i=1,2)$ has the following pdf:

$$g_i(s_i^2)=\left(\frac{m}{2\sigma_i^2}\right)^{m/2}\frac{(s_i^2)^{(m-2)/2}}{\Gamma(m/2)}\exp\left(\frac{-ms_i^2}{2\sigma_i^2}\right),\quad (i=1,2).$$

Proof. The proof requires the evaluation of the following integral :

$$g_i(s_i^2)=\int_0^{\infty}f_2(s_1^2,s_2^2)ds_i^2,\ \ i=1,2,$$

where $f_2(s_1^2,s_2^2)$ is defined in (3.1).

It can be checked that $mS_i^2/\sigma_i^2 \sim \chi_m^2,\ i=1,2.$

Theorem 4.2 Let S_1^2 and S_2^2 be two correlated sample variances with joint pdf given by (3.1). Then the conditional pdf of S_1^2 given $S_2^2=s_2^2$ is given by

$$f_2(s_1^2\mid s_2^2)=\left(\frac{m}{(2-2\rho^2)\sigma_1^2}\right)^{m/2}\frac{s_1^{m-2}}{\Gamma(m/2)}\exp\left[\frac{-m}{2(1-\rho^2)}\left(\frac{s_1^2}{\sigma_1^2}+\frac{\rho^2s_2^2}{\sigma_2^2}\right)\right] \times {}_0F_1\left(\frac{m}{2};\frac{(m\rho)^2s_1^2s_2^2}{(2-2\rho^2)^2\sigma_1^2\sigma_2^2}\right). \tag{4.1}$$

where $m>2,\ -1<\rho<1$ and ${}_0F_1(a;z)$ is defined by (2.1).

Proof. The theorem follows from

$$f_2(s_1^2 \mid s_2^2) = \frac{f_2(s_1^2, s_2^2)}{g_2(s_2^2)}$$

where $g_2(s_2^2)$ is given by Theorem 4.1

Theorem 4.3 Le S_1^2 and S_2^2 be correlated sample variances with joint pdf given by (3.1). Then the regression function of S_1^2 given $S_2^2 = s_2^2$ is given by

$$E(S_1^2 \mid S_2^2 = s_2^2) = [m(1-\rho^2) + \rho^2 v]\frac{\sigma_1^2}{m}, \; m > 2, \; -1 < \rho < 1.$$

Proof. The regression function of S_1^2 given $S_2^2 = s_2^2$ follows from the definition

$$E(S_1^2 \mid S_2^2 = s_2^2) = \int_0^\infty s_1^2 f_2(s_1^2 \mid s_2^2) ds_1^2.$$

Theorem 4.4 Le S_1^2 and S_2^2 be correlated sample variances with joint pdf given by (3.1). Then the a-th moment of the conditional distribution S_1^2 given $S_2^2 = s_2^2$ is given by

$$E(S_1^{2a} \mid S_2^2 = s_2^2) = \left(\frac{(2-2\rho^2)\sigma_1^2}{m}\right)^a \frac{\Gamma\left(a + (m/2)\right)}{\Gamma\left(m/2\right)} e^{-z} {}_1F_1\left(a + \frac{m}{2}; \frac{m}{2}; z\right), \tag{4.2}$$

where $m > 2, -1 < \rho < 1$ and ${}_1F_1(a;b;z)$ is defined by (2.1) and (2.3).

Proof. The regression function of S_1^2 given $S_2^2 = s_2^2$ follows from the definition

$$E(S_1^{2a} \mid S_2^2 = s_2^2) = \int_0^\infty s_1^{2a} f_2(s_1^2 \mid s_2^2) ds_1^2.$$

The derivation requires evaluation of a gamma integral, and simplification of hypergeometric function.

Corollary 4.1 Le S_1^2 and S_2^2 be correlated sample variances with joint pdf given by (3.1). Then for any positive integer a, the a-th moment of the conditional distribution of S_1^2 given S_2^2 has the following representations:

$$E\left(S_1^{2a} \mid S_2^2 = s_2^2\right) = \left(\frac{\sigma_1^2}{m}\right)^a \sum_{k=0}^{a} \binom{a}{k} \left(\frac{m s_2^2}{\sigma_2^2}\rho^2\right)^k (2-2\rho^2)^{a-k} \left(m/2\right)_{\{a-k\}}, \tag{4.3}$$

or, $$E(S_1^{2a} \mid S_2^2 = s_2^2) = \left(\frac{\sigma_1^2}{m}\right)^a (2-2\rho^2)^a a! L_a^{(m-2)/2}(-z), \tag{4.4}$$

where $-1 < \rho < 1$, $m > 2$, $L_n^{(\alpha)}(x)$ is the generalized Laguerre Polynomial defined by (2.2) and z is defined by $(2-2\rho^2)\sigma_2^2 z = ms_2^2\rho^2$.

Proof. By using (2.1) in (2.3), we have

$$ {}_1F_1\left(a+(m/2);(m/2);z\right) = e^z \sum_{k=0}^{\infty} \frac{(-a)_{\{k\}}}{(m/2)_{\{k\}}} \frac{(-z)^k}{k!},$$

which, by virtue of (2.6) and (2.7), can be expressed as

$$ {}_1F_1\left(a+(m/2);(m/2);z\right) = e^z \sum_{k=0}^{a} \frac{a^{\{k\}}}{(m/2)_{\{k\}}} \frac{z^k}{k!}.$$

Since $\binom{a}{k} = \frac{a^{\{k\}}}{k!}$, we have

$$ {}_1F_1\left(a+(m/2);(m/2);z\right) = e^z \sum_{k=0}^{a} \binom{a}{k} \frac{z^k}{(m/2)_{\{k\}}}. \tag{4.5}$$

Hence by using (4.5) in (4.2), we have (4.3).

Next, by using (2.3) in (4.2), we have

$$E(S_1^{2a} \mid S_2^2) = \frac{(2-2\rho^2)^a}{\Gamma(m/2)} \Gamma\left(a+(m/2)\right) {}_1F_1\left(-a;(m/2);-z\right). \tag{4.6}$$

Putting $n = a, \alpha + 1 = (m/2)$ in (2.2), we have

$$L_a^{(m-2)/2}(-z) = \frac{\Gamma\left(a+(m/2)\right)}{a!\Gamma\left(m/2\right)} {}_1F_1\left(-a;(m/2);-z\right). \tag{4.7}$$

By using (4.7) in (4.6), we have (4.4).

Corollary 4.2 The second, third and fourth order moments of the conditional pdf of sample variances are given by

$$E(S_1^4 \mid S_2^2 = s_2^2) = \left[\rho^4 v^2 + 2(m+2)\rho^2 v(1-\rho^2) + m(m+2)(1-\rho^2)^2\right] \frac{\sigma_1^4}{m^2}, \tag{4.8}$$

$$E\left[\left(S_1^2\right)^3 \middle| s_2^2\right] = \left[\rho^6 v^3 + 3(m+4)\rho^4(1-\rho^2)v^2 + 3(m+2)(m+4)\rho^2(1-\rho^2)^2 v + m(m+2)(m+4)(1-\rho^2)^3\right]\left(\sigma_1^2 / m\right)^3, \quad (4.9)$$

and

$$E\left[\left(S_1^2\right)^4 \middle| s_2^2\right] = \left[\rho^8 v^4 + 4(m+6)\rho^6(1-\rho^2)v^3 + 6(m+4)(m+6)(1-\rho^2)^2 v^2 + 4(m+2)(m+4)(m+6)\rho^2(1-\rho^2)^3 v + M(1-\rho^2)^4\right]\left(\sigma_1^2 / m\right)^4, \quad (4.10)$$

respectively where $M = m(m+2)(m+4)(m+6)$ and $v = ms_2^2 / \sigma_2^2$.

Corollary 4.3 The second, third and fourth order corrected moments of the conditional pdf of sample variances are given by

$$E\left[\left\{S_1^2 - E\left(S_1^2 \mid S_2^2\right)^2\right\} \mid S_2^2\right] = \left[2m\left(1-\rho^2\right)^2 + 4\rho^2\left(1-\rho^2\right)\frac{ms_2^2}{\sigma_2^2}\right]\left(\sigma_1^2 / m\right)^2, \quad (4.11)$$

$$E\left[\left\{S_1^2 - E\left(S_1^2 \mid S_2^2\right)^3\right\} \mid S_2^2\right] = 8\left(1-\rho^2\right)^2\left[m\left(1-\rho^2\right) + \frac{3ms_2^2\rho^2}{\sigma_2^2}\right]\left(\sigma_1^2 / m\right)^3, \quad (4.12)$$

and

$$E\left[\left\{S_1^2 - E\left(S_1^2 \mid S_2^2\right)\right\}^4 \mid S_2^2\right] = 12\left(1-\rho^2\right)^2\left[4\rho^4 v^2 + 4(m+4)\rho^2\left(1-\rho^2\right)v + m(m+4)\left(1-\rho^2\right)^2\right]\left(\sigma_1^2 / m\right)^4, \quad (4.13)$$

respectively where $m > 2$ and $-1 < \rho < 1$.

The moment in (4.11) is the variance of the conditional pdf of variances and is popularly denoted by $Var(S_1^2 \mid S_2^2 = s_2^2)$.

Corollary 4.4 The coefficient of skewness of the conditional distribution of variances is given by

$$\frac{E\left[\left\{S_1^2 - E\left(S_1^2 \mid S_2^2\right)\right\}^3 \mid S_1^2\right]}{\left[Var\left(S_1^2 \mid S_2^2\right)\right]^{3/2}} = \frac{\sqrt{8\left(1-\rho^2\right)}\left[m\left(1-\rho^2\right) + 3\rho^2 ms_2^2 / \sigma_2^2\right]}{\left[m\left(1-\rho^2\right) + 2\rho^2 ms_2^2 / \sigma_2^2\right]^{3/2}}. \quad (4.14)$$

If $\rho = 0$, the coefficient of skewness reduces to $\sqrt{8/m}$, which is the coefficient of skewness for univariate chi-square distribution with m degrees of freedom (See Johnson, Kotz and Balakrishnan, 1994, 420). If $\rho = 0$ and m increases indefinitely, then the coefficient of skewness of the conditional distribution would converge, as expected, to 0.

Corollary 4.5 The coefficient of kurtosis of the conditional distribution of variances is given by

$$\frac{E[\{(S_1^2 - E(S_1^2 \mid s_2^2)\}^4 \mid s_2^2]}{[Var(S_1^2 \mid s_2^2)]^2} = \frac{3[4\rho^4 v^2 + 4(m+4)\rho^2(1-\rho^2)v + m(m+4)(1-\rho^2)^2]}{[2\rho^2 v + m(1-\rho^2)]^2}, \qquad (4.15)$$

where $v = ms_2^2 / \sigma_2^2$, $-1 < \rho < 1$, and $m > 2$. If $\rho = 0$, the coefficient of kurtosis reduces to $3 + 12/m$, which is the coefficient of kurtosis for univariate chi-square distribution with m degrees of freedom (See Johnson, Kotz and Balakrishnan, 1994, 420). If $\rho = 0$ and m increases indefinitely, then the coefficient of kurtosis of the conditional distribution would converge, as expected, to 3.

5. Product Moments

The following theorem reported in Joarder and Abujiya (2008) and Joarder (2009) has been expressed in generalized hypergeometric and some other functions in this section.

Theorem 5.1 Let the sample variances S_1^2 and S_2^2 have the joint pdf in (3.1). Then for $a > -m/2,\ b > -m/2$ and $-1 < \rho < 1$, the (a,b)-th product moment of S_1^2 and S_2^2, denoted by $\mu'(a,b;\rho) = E(S_1^{2a} S_2^{2b})$, is given by

$$\mu'(a,b;\rho) = \left(\frac{2}{m}\right)^{a+b} \sigma_1^{2a}\sigma_2^{2b} \frac{\Gamma\left((m/2)+a\right)\Gamma\left((m/2)+b\right)}{\Gamma^2(m/2)} \, {}_2F_1\left(-a,-b;(m/2);\rho^2\right). \qquad (5.1)$$

Proof. The (a,b)-th product moment $\mu'(a,b;\rho) = E(S_1^{2a} S_2^{2b})$ is given by

$$\mu'(a,b) = \frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}} \frac{2^{a+b}(1-\rho^2)^{a+b+(m/2)}}{\sqrt{\pi}\,\Gamma(m/2)} \sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{(2k)!} \frac{\Gamma\left(k+a+(m/2)\right)}{\Gamma\left(k+(m/2)\right)} \Gamma\left(k+b+(m/2)\right)\Gamma\left(k+\tfrac{1}{2}\right).$$

Since $(2z)!\sqrt{\pi} = 2^{2z} z!\,\Gamma\left(z+(1/2)\right)$, the above can be written as

$$\mu'(a,b;\rho)=\frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}}\frac{2^{a+b}(1-\rho^2)^{a+b+(m/2)}}{\Gamma(m/2)}\sum_{k=0}^{\infty}\frac{\rho^{2k}}{k!}\frac{\Gamma\left(k+a+(m/2)\right)}{\Gamma\left(k+(m/2)\right)}\Gamma\left(k+b+(m/2)\right). \quad (5.2)$$

Then by hypergeometric function (2.1), we have

$$\mu'(a,b;\rho)=\frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}}2^{a+b}(1-\rho^2)^{a+b+(m/2)}\frac{\Gamma\left(a+(m/2)\right)\Gamma\left(b+(m/2)\right)}{\Gamma^2(m/2)}$$
$$\times\,{}_2F_1\left(a+(m/2),b+(m/2);(m/2);\rho^2\right)$$

which can be transformed to (5.1) by virtue of (2.4).

Corollary 5.1 Let the sample variances S_1^2 and S_2^2 have the joint pdf given by (3.1). Then for nonnegative integers a and b, $m>2$ and $-1<\rho<1$, we have the following:

$$(i)\ \mu'(a,b;\rho)=\frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}}2^{a+b}(m/2)_{\{b\}}\sum_{k=0}^{a}(-1)^k\binom{a}{k}(1+b-k)_{\{k\}}\left((m+2k)/2\right)_{\{a-k\}}\rho^{2k}$$
,

$$(ii)\ \mu'(a,a;\rho)=\frac{\left(\sigma_1^2\sigma_2^2\right)^a}{m^{2a}}\ 4^a(m/2)_{\{a\}}\sum_{k=0}^{a}\binom{a}{k}(1+a-k)_{\{k\}}\left((m+2k)/2\right)_{\{a-k\}}\rho^{2k},$$

$$(iii)\ \mu'(a,-a;\rho)=\left(\frac{\sigma_1^2}{\sigma_2^2}\right)^a\ \frac{(m/2)_{\{a\}}}{\left((m-2)/2\right)^{\{a\}}}\sum_{k=0}^{a}\binom{a}{k}\frac{(1-a-k)_{\{k\}}}{(m/2)_{\{k\}}}\rho^{2k},$$

where $a^{\{k\}}$ and $a_{\{k\}}$ are defined by (2.6) and (2.7) respectively.

Proof. (i) Since ${}_2F_1(-a,-b;(m/2);\rho^2)=\sum_{k=0}^{a}\frac{(-a)_{\{k\}}(-b)_{\{k\}}}{(m/2)_{\{k\}}}\frac{\rho^k}{k!}$, (See 15.4.1 of Abramowitz and Stegun, 1972, p561), from Theorem 5.1, we have

$$\mu'(a,b;\rho)=2^{a+b}\frac{\Gamma\left(a+(m/2)\right)\Gamma\left(b+(m/2)\right)}{\Gamma^2(m/2)}\sum_{k=0}^{a}\frac{(-a)_{\{k\}}(-b)_{\{k\}}}{(m/2)_{\{k\}}}\frac{\rho^k}{k!}.$$

Further by virtue of $(-a)_{\{k\}}\Gamma(a-k+1)=(-1)^k\ \Gamma(a+1)$, we have

$$\mu'(a,b;\rho)=\frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}}2^{a+b}\frac{\Gamma\left(a+(m/2)\right)\Gamma\left(b+(m/2)\right)}{\Gamma\left(m/2\right)\Gamma\left(m/2\right)}\sum_{k=0}^{a}\binom{a}{k}\frac{(b-k+1)_{\{k\}}\Gamma\left(m/2\right)}{\Gamma\left((m/2)+k\right)}\rho^{2k},$$

which is equivalent to (i), since $a_{\{k\}}\Gamma(a)=\Gamma(a+k)$.

(ii) Putting a for b in Theorem 5.1, we have

$$\mu'(a,a)=\frac{\sigma_1^{2a}\sigma_2^{2a}}{m^{a+a}}2^{2a}\frac{\Gamma^2\left(a+(m/2)\right)}{\Gamma^2(m/2)}\,{}_2F_1\left(-a,-a;(m/2);\rho^2\right)$$

which is the same as what we have in part (ii) of the corollary.

(iii) Putting $-a$ for b in Theorem 5.1, we have

$$\mu'(a,-a;\rho)=\frac{\sigma_1^{2a}\sigma_2^{-2a}}{m^{a-a}}\frac{\Gamma\left(a+(m/2)\right)\Gamma\left(-a+(m/2)\right)}{\Gamma(m/2)\Gamma(m/2)}\sum_{k=0}^{a}\frac{a!(-a-k+1)_{\{k\}}}{(a-k)!(m/2)_{\{k\}}}\frac{\rho^{2k}}{k!}.$$

Since by (2.6), $a^{\{k\}}\Gamma(a+1-k)=\Gamma(a+1)$, the above can be written as what we have in part (iii) of the corollary.

The moments in (ii) and (iii) represent the a-th moment of $S_1^2S_2^2$ and S_1^2/S_2^2 respectively whenever a is a nonnegative integer. The above moments in (i) and (ii) are represented by Jacobi's Polynomials in the following corollary:

Corollary 5.2 Let the sample variances S_1^2 and S_2^2 have the joint pdf given by (3.1). Then for nonnegative integers a and b, $m>2$, and $-1<\rho<1$, we have the following:

$$(i)\ \mu'(a,b;\rho)=\frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}}\ 2^{a+b}\left(m/2\right)_{\{b\}}(1-\rho^2)^{-a-b}a!P_a^{((m/2)-1,-a-b-(m/2))}(1-2\rho^2),$$

$$(ii)\ \mu'(a,a;\rho)=\frac{\left(\sigma_1^2\sigma_2^2\right)^a}{m^{2a}}\ 4^a\left(m/2\right)_{\{a\}}(1-\rho^2)^{-2a}a!P_a^{((m/2)-1,-2a-(m/2))}(1-2\rho^2).$$

Proof. Putting $a=n,\ \alpha+1+\beta+a=-b,\ \alpha+1=(m/2),\ z=\rho^2$ in (2.5), we have

$${}_2F_1(-a,-b;(m/2);\rho^2)=\frac{a!}{(m/2)_{\{a\}}}P_a^{((m/2)-1,-a-b-(m/2))}(1-2\rho^2).$$

Then (i) and (ii) follows from Theorem 5.1.

The corrected product moments of the joint pdf of sample variances can also be calculated by $\mu(a,b;\rho)=E[(S_1^2-E(S_1^2))^a(S_2^2-E(S_2^2))^b]$ with the help of $\mu'(a,b;\rho)$. Then one can write out the covariance matrix of bivariate chi-square distribution by

$$Var\left(\frac{mS_1^2}{\sigma_1^2}\right)=2m,\ Var\left(S_1^2\right)=\frac{2\sigma_1^4}{m},\ Var\left(S_2^2\right)=\frac{2\sigma_2^4}{m},\ \text{and}\ \ Cov\left(S_1^2,S_2^2\right)=\frac{2\sigma_1^2\sigma_2^2\rho^2}{m},$$

which also clearly shows that the correlation between U and V is ρ^2.

If $\rho = 0$ in (3.1), $Cor(S_1^2, S_2^2) = \rho^2 = 0$, that is, if sample variances are uncorrelated, it can be checked that $f_2(s_1^2, s_2^2) = g_1(s_1^2)\, g_2(s_2^2)$. That is, the joint pdf in (3.1), is an example that exhibits that uncorrelation of S_1^2 and S_2^2 implies independence between them.

6. Shannon Entropy

The Shannon Entropy $H(f)$ for any bivariate density function $f(x_1, x_2)$ is defined by $H(f) = -E\left(\ln f(X_1, X_2)\right)$. For the bivariate normal distribution it is given by $H(f) = \ln(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}) + 1$. In the following theorem we derive the Shannon entropy for the joint pdf of sample variances.

Theorem 6.1 Let the sample variances S_1^2 and S_2^2 have the joint pdf of in (3.1). Then the Shannon entropy $H(f)$ of the distribution is given by

$$-H(f) = (m-2)\left[\log 2 + \psi(m/2)\right] - \ln\left[2^m\ \Gamma^2(m/2)(1-\rho^2)^{m/2}\right] - \frac{m}{1-\rho^2} + E\left[\ln\ {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 m^2 S_1^2 S_2^2}{\sigma_1^2\sigma_2^2(2-2\rho^2)^2}\right)\right], \tag{6.1}$$

where $\psi(z) = \dfrac{d}{dz}\ln\Gamma(z)$ is a digamma function.

Proof. From (3.1), the Shannon entropy $H(f) = -E\left(\ln f(S_1^2, S_2^2)\right)$ is given by

$$-H(f) = E\left[\ln\frac{(S_1^2 S_2^2)^{(m-2)/2}}{2^m\ \Gamma^2(m/2)(1-\rho^2)^{m/2}} - \frac{S_1^2 + S_2^2}{2-2\rho^2} + \ln\ {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 S_1^2 S_2^2}{(2-2\rho^2)^2}\right)\right]$$

which simplifies to

$$-H(f) = \frac{m-2}{2}\left[E(\ln S_1^2) + E(\ln S_2^2)\right] - \ln\left[2^m\ \Gamma^2(m/2)(1-\rho^2)^{m/2}\right] - \frac{m}{1-\rho^2} + E\left[\ln\ {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 S_1^2 S_2^2}{(2-2\rho^2)^2}\right)\right].$$

Since S_1^2 or S_2^2 has a chi-square distribution with m degrees of freedom, the theorem is then obvious by virtue of $E(S_1^2) = \frac{\sigma_1^2}{m}[\log 2 + \psi(m/2)]$ where $\psi(z)$ is the digamma function defined in the theorem.

7. Conclusion

We have derived some characteristics of joint distribution of variances which will have applications in the estimation of generalized variance and other inferential methods. It is also open to extend these results to any other bivariate distribution especially to scale mixture of bivariate normal distribution which includes bivariate t-distribution.

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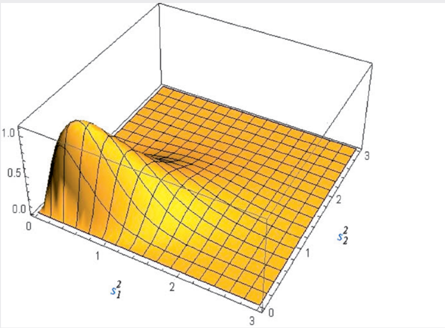
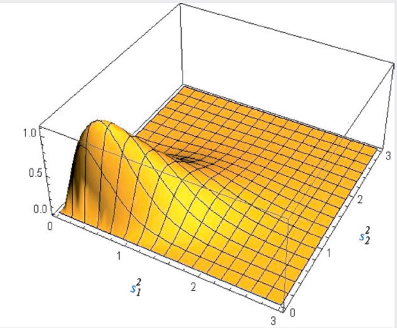
Appendix



a) ρ = -0.8



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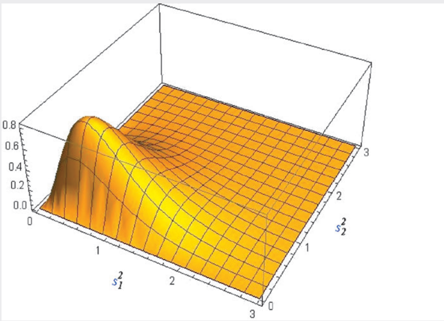
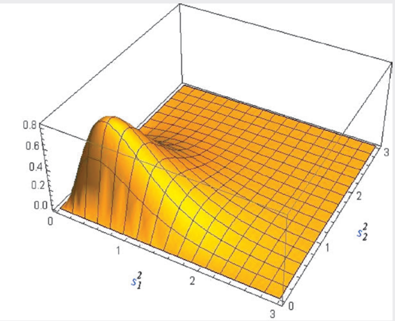
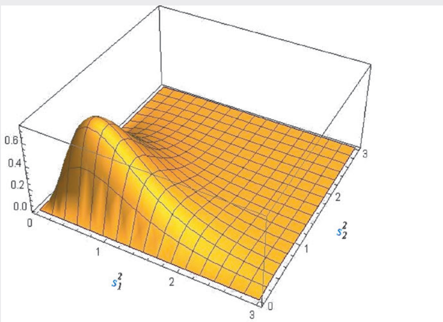


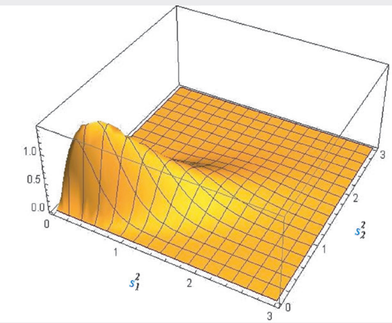
c) ρ = -0.5



d) ρ = 0.5



e) ρ = 0



f) ρ = 0.9

Figure 1. Joint Probability Density Function of Sample Variances for $m = 5$, $\sigma_1 = 1$, $\sigma_2 = 0.9$, and various ρ values